Takahiro Tanaka
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

There are several works searching for a clue of trans-Planckian physics on the primordial density perturbation spectrum. Here we would like to point out an important aspect which has been overlooked so far. When we consider a model in which the primordial density perturbation spectrum is modified due to trans-Planckian physics, the energy density of fluctuations of the inflaton field necessarily becomes significantly large, and hence its back reaction to the cosmic expansion rate cannot be neglected.

In quantum field theory in curved space we usually decompose the field operator by using mode functions, which are the appropriately normalized solutions of the field equation. Many phenomena in early universe and radiation in black hole spacetime have been discussed based on this effective theory. In some cases, mode functions which had an infinitesimally short wave length are redshifted by an infinitely large amount, and become relevant modes which contribute to observable effects. In such cases, a simple question arises: Does any observable effect appear if we assume a certain modification for trans-Planckian physics? There are at least two cases in which context this question has been discussed. One is the Hawking radiation [1–10] and the other is the quantum dynamics of a scalar field in an expanding universe, especially in the context of the inflationary universe scenario scenario [11–14]. (For a review of the general issue as to trans-Planckian redshifts in cosmology and black hole physics, See Ref. [15] and references therein.)

In the former context, people studied models with a modified dispersion relation in most cases so far. Basically they reported that the thermal spectrum of Hawking radiation is reproduced in spite of introduction of a modified law of trans-Planckian physics. In the latter context, this issue is discussed only recently, in which they again examined models with a modified dispersion relation. The main question in the latter studies is whether a modification of trans-Planckian physics can cause a non-scale invariant spectrum of primordial density fluctuations. However, there seems to be an aspect which has been overlooked so far, but which is rather important in discussing trans-Planckian physics consistently in the context of the inflationary universe scenario.

In the standard inflation scenario, the quantum state of the inflaton field is thought to be set almost to be in a "vacuum" initially at a time after the wavelength of each mode becomes sufficiently long compared with the Planck scale. As is known well, the concept of the vacuum state is quite ambiguous in curved space. In the context of the inflationary universe scenario, however, this ambiguity can be ignored by choosing the adiabatic vacuum when the wavelength of a mode is much shorter than the curvature scale of spacetime. This prescription applies in the standard inflationary universe scenario. Together with this assumption as to the initial condition, we also assume that the quantum field theory in curved space is valid after the wavelength becomes much longer than the Planck scale. As a result of these assumptions, we usually conclude the appearance of an almost scale invariant spectrum for initial fluctuations.

Even when we assume a non-trivial trans-Planckian physics, we suppose that physics after the wavelength becomes much longer than the Planck scale can be described by the standard quantum field theory in curved space. If this is not the case, the modification should not be referred to that of trans-Planckian physics. Thus, if we discuss the evolution of each mode after its wavelength becomes much longer than the Planck scale, i.e., if we restrict our attention to the regime in which the standard quantum field theory in curved space applies, the effect of a modified Planck scale physics comes into play only through the initial condition. Hence, in order to construct a model which leads to results different from the standard prediction, the quantum state of the inflaton field must be chosen differently from the ordinary vacuum as the initial condition at the very beginning. In this sense, a non-trivial initial condition is necessarily assumed when we study trans-Planckian physics in the cosmological context.

However, setting a non-trivial initial quantum state for the inflaton field means, in some sense, that we consider a situation in which there exist a significant amount of inflaton particles. Even if we restrict our consideration to the modes whose wavelength has already become much longer than the Planck length, there are a lot of particles which have a relatively high momentum in comparison with the Hubble scale. The energy density of these inflaton particles will be given by $\int dp\, p^3 n_p$ denoting the occupation number of the modes with momentum p as n_p. As we will see later, n_p cannot be much smaller than unity in order to obtain a significantly modified initial power spectrum. Since we assumed that the physics is modified only at the Planck scale M_{pl}, it will be natural to suppose that the cutoff scale of p below which the energy density can be well approximated by the above expression is not much smaller than M_{pl}. Hence the cutoff scale is much greater than $\sqrt{M_{pl}H}$, where H is the expansion rate of the universe. Then we find that the energy density of these inflaton particles dominates over the vacuum energy density ($\approx M_{pl}^2 H^2$) which is supposed to drive the universe to expand.

To make the above argument more precise, we would like to present a more explicit calculation briefly. For simplicity, we consider a flat de Sitter model, whose metric is given by

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + d\mathbf{x}^2\right). \tag{1}$$

We treat here fluctuations of the inflaton field as a massless scalar field. As usual, the fluctuation field will be decomposed by using mode functions $u_{\mathbf{k}}(x)$ as

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}}\left(a_{\mathbf{k}} u_{\mathbf{k}}(x) + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(x)\right), \tag{2}$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ are annihilation and creation operators, respectively. The natural adiabatic vacuum state, i.e., so-called Bunch-Davies vacuum state, is specified by $a_{\mathbf{k}}|0_a\rangle = 0$ with

$$u_{\mathbf{k}} = \frac{H\eta\, e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3}$$

Here we consider a situation in which the quantum state of fluctuations of the inflaton field is in another vacuum state specified by

$$b_{\mathbf{k}}|0_b\rangle = 0, \tag{4}$$

with

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}}\left(b_{\mathbf{k}}v_{\mathbf{k}}(x) + b_{\mathbf{k}}^{\dagger}v_{\mathbf{k}}^{*}(x)\right), \tag{5}$$

and

$$v_{\mathbf{k}} = \alpha_{\mathbf{k}}u_{\mathbf{k}} + \beta_{\mathbf{k}}u_{\mathbf{k}}^{*}. \tag{6}$$

Assuming this b-vacuum state, we compute the amplitude of a fluctuation mode at a late epoch after its wavelength becomes much longer than the Horizon scale as

$$\langle 0_b|\,|\phi_{\mathbf{k}}|^2\,|0_b\rangle \xrightarrow{\eta\to 0} \frac{H^2}{2k^3}\left(|\alpha_{\mathbf{k}}|^2 + |\beta_{\mathbf{k}}|^2 + 2|\alpha_{\mathbf{k}}|\cdot|\beta_{\mathbf{k}}|\cos\theta\right), \tag{7}$$

where we have introduced $\theta = \arg(\alpha_{\mathbf{k}}) - \arg(\beta_{\mathbf{k}})$. Using the well-known relation $|\alpha_{\mathbf{k}}|^2 = 1 + |\beta_{\mathbf{k}}|^2$, this amplitude of a fluctuation is found to be bounded from above as

$$\langle 0_b|\,|\phi_{\mathbf{k}}|^2\,|0_b\rangle < \frac{H^2}{2k^3}\left(\sqrt{1+|\beta_{\mathbf{k}}|^2} + |\beta_{\mathbf{k}}|\right)^2. \tag{8}$$

From this, we can conclude that the occupation number $n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2$ cannot be much smaller than unity under the condition that $\langle 0_b|\,|\phi_{\mathbf{k}}|^2\,|0_b\rangle$ significantly differs from $H^2/2k^3$, the value corresponding to the Bunch-Davies vacuum.

On the other hand, the expectation value of the energy momentum tensor, $T_{\mu\nu} = \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\phi_{;\rho}\phi_{;\sigma}$, will be given by

$$\begin{aligned}
\langle 0_b|\,(-T^0_0)\,|0_b\rangle &= \frac{H^2\eta^2}{2}\int\frac{d^3k}{(2\pi)^3} \\
&\quad\times\Bigg[(\alpha_{\mathbf{k}}u'_{\mathbf{k}} + \beta_{\mathbf{k}}u'^{*}_{\mathbf{k}})(\alpha^{*}_{\mathbf{k}}u'^{*}_{\mathbf{k}} + \beta^{*}_{\mathbf{k}}u'_{\mathbf{k}}) \\
&\qquad + \mathbf{k}^2(\alpha_{\mathbf{k}}u_{\mathbf{k}} - \beta_{\mathbf{k}}u^{*}_{\mathbf{k}})(\alpha^{*}_{\mathbf{k}}u^{*}_{\mathbf{k}} - \beta^{*}_{\mathbf{k}}u_{\mathbf{k}})\Bigg] \\
&\xrightarrow{\eta\to-\infty} \frac{(H\eta)^4}{2(2\pi)^3}\int d^3k\cdot k\Bigg[|\alpha_{\mathbf{k}}|^2 + |\beta_{\mathbf{k}}|^2 \\
&\qquad - (\alpha_{\mathbf{k}}\beta^{*}_{\mathbf{k}}e^{-2ik\eta} + \alpha^{*}_{\mathbf{k}}\beta_{\mathbf{k}}e^{2ik\eta})\Bigg].
\end{aligned} \tag{9}$$

As is known well, this expression is divergent for any choice of $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$. To obtain a finite result, it is necessary to implement some renormalization procedure which subtract appropriate covariant counter terms written in terms of geometrical quantities [16]. Here we adopt a simple prescription in which we just subtract the expectation value of the energy momentum tensor for the Bunch-Davies vacuum, and we refer to it as a renormalized quantity. Namely, we calculate $\langle 0_b|\,(-T^0_0)\,|0_b\rangle^{(ren)} \equiv \langle 0_b|\,(-T^0_0)\,|0_b\rangle - \langle 0_a|\,(-T^0_0)\,|0_a\rangle$. To remove the highly oscillatory contribution from the terms in the round brackets in the last line of Eq.(9), we consider a time averaged quantity. Then we find that the energy density due to the inflaton particles is given by

$$\begin{aligned}
\frac{1}{\Delta\eta}\int_{\eta-\Delta\eta/2}^{\eta+\Delta\eta/2} d\eta'\langle(-T^0_0)\rangle^{(ren)} &\approx \frac{(H\eta)^4}{(2\pi)^3}\int d^3k\cdot k|\beta_{\mathbf{k}}|^2 \\
&= \frac{1}{(2\pi)^3}\int d^3p\cdot p\,n_p,
\end{aligned} \tag{10}$$

where we have introduced the physical momentum $p \equiv k/a$. Now we have obtained the anticipated result.

As we have mentioned earlier, this result tells us that the energy density due to fluctuations of the inflaton field becomes larger than that due to the inflaton potential as long as the cutoff scale for momentum below which the ordinary quantum field theory in curved space holds is chosen to be greater than $\sqrt{M_{pl}H}$ or equivalently $V^{1/4}$, where V is the energy density due to the inflaton potential. When we discuss the effect of trans-Planckian physics, this cutoff scale would be supposed to be near the Planck scale. In such cases, the basic assumption as to the background inflationary universe will not be appropriate because the back reaction becomes significant. Here we mention that the order of magnitude of Eq.(10) does not change even if we consider the case in which $n_{\mathbf{k}}$ is non-trivial only for fluctuations in a rather narrow band of k.

If we wish to construct a consistent model with a large value of $n_{\mathbf{k}}$, we need to invent a certain mechanism of generating negative energy density which cancels a large amount of energy density due to fluctuations of the inflaton field. Moreover, even if we could construct a model with such a mechanism, we need to suppose that $n_{\mathbf{k}}$ is not constant in k in order to realize a non-trivial initial power spectrum. Then, it is easy to see from Eq.(10) that the energy density due to fluctuations becomes time-dependent. Again, if we consider the case in which $n_{\mathbf{k}}$ is non-trivial only in a narrow band of k, the energy density due to fluctuations behaves as a radiation field, i.e, $\propto a^{-4}$. Therefore, it is not clear whether the background accelerated expansion of the universe is realized.

Furthermore, it seems that we also need to pay attention to the fact that non-trivial vacuum means existence of particles in some sense, although we did not discuss it in this short report. Under the circumstance filled with a lot of particles with high momentum, interaction between particles should be taken into account.

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